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Exponential Mixing for the Geodesic Flow on Hyperbolic Three-Manifolds

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We give a short and direct proof of exponential mixing of geodesic flows on compact hyperbolic three-manifolds with respect to the Liouville measure. This complements earlier results of Collet–Epstein–Gallovotti, Moore, and Ratner for hyperbolic surfaces. Furthermore, since the analysis is even easier in three dimensions than in two dimensions (because of the absence of discrete series and the simplicity of the zonal spherical functions in this case), this apparently gives the simplest example of a flow with exponential mixing.

KEY WORDS: Mixing; correlation; exponential; geodesic; representation.

INTRODUCTION

In recent years several authors have studied the rate of mixing for geodesic flows on compact hyperbolic manifolds (i.e., manifolds with constant sectional curvatures) with respect to the Liouville measure. These are now the basic examples of flows which exhibit exponential mixing (or "exponential decay of correlations").

Not surprisingly, the case which has received most attention is that of hyperbolic surfaces. The first result in this direction was apparently the 1984 paper of Collet *et al.*,[2] where they showed exponential mixing for geodesic flows on hyperbolic surfaces (and test functions constant on the sphere bundle)—modulo a minor technical omission in their proof. Subsequently, Moore[3] showed how very general notions in representation theory can be used to deal with the surface case, and sketched how they could be extended to hyperbolic manifolds of arbitrary dimension (and other locally symmetric manifolds). Finally, Ratner[5] presented yet another

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alternative analysis of the surface case (which differed from the previous two in that it required a little less representation theory).

In this paper we shall consider the case of geodesic flows on three-dimensional hyperbolic manifolds and test functions which are constant on the fibers of the sphere bundle. *The most interesting point is that the analysis for three dimensions turns out to be even easier than for surfaces* and thus we have what is apparently the *easiest* example of exponential mixing for flows. The analysis is closest in spirit to that of ref. 1 in that we shall use the classification of the irreducible unitary representations of the associated Lie group $SL(2, \mathbb{C})$.

1. GEODESIC FLOWS AND FRAME FLOWS

For a compact hyperbolic three-manifold V it is well known that we can represent the (five-dimensional) unit tangent bundle SV algebraically as $\Gamma \backslash G/B$, where:

(i) $G = PSL(2, \mathbb{C})$ is the group of 2×2 complex unimodular matrices.

(ii) Γ is a discrete subgroup of G (called a *Kleinian group*).

(iii)

$$B = \left\{ \begin{bmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{bmatrix} : \quad 0 \leqslant t \leqslant 2\pi \right\}$$

is a compact subgroup (called the *Borel* subgroup).

We shall be interested in the one-parameter subgroup

$$g_t = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{bmatrix} \qquad (t \in \mathbb{R})$$

Definition. The *frame flow* ψ_t: $\Gamma \backslash G \to \Gamma \backslash G$ corresponds to the action of the one-parameter subgroup g_t $(t \in \mathbb{R})$ defined on the cosets by $\Gamma g \mapsto \Gamma g g_t$. The *geodesic flow* ϕ_t: $\Gamma \backslash G/B \to \Gamma \backslash G/B$ corresponds to the action on the double cosets by $\Gamma g B \mapsto \Gamma g g_t B$.

Geometrically, $\Gamma \backslash G$ corresponds to the two-frame bundle over V, consisting of a distinguished choice of unit tangent vector and a subsequent choice of orthonormal vector. If we consider the maximal compact subgroup

$$K = SU(2) = \left\{ -i \begin{bmatrix} -\bar{z}_2 & \bar{z}_1 \\ z_1 & z_2 \end{bmatrix} \subset G : z_1, z_2 \in \mathbb{C}, \ |z_1|^2 + |z_2|^2 = 1 \right\}$$

then its action corresponds to rotating the two-frames in each fiber of the sphere bundle over V and therefore the quotient $\Gamma\backslash G/K$ can be identified with the manifold V. The Borel subgroup B is a subgroup of the maximal compact subgroup K and the geometric interpretation of the action of B on the two-frames corresponds to fixing the distinguished unit tangent vector and rotating the orthonormal vector. In particular, we can identify the double quotient space $\Gamma\backslash G/B$ with the unit tangent bundle SV of V as claimed above.

The one-parameter group g_t on $\Gamma\backslash G$ defines the frame flow on the two-frame bundle over the manifold. Geometrically, the distinguished tangent vector determines a geodesic on V and is parallel-transported for time t along the geodesic, as is the orthonormal vector. This frame flow is thus a compact group extension of the geodesic flow by $SO(2)$. Since the actions of B and g_t commute, we can quotient out by the action of B, to remove the choice of orthonormal vector, and this leaves the geodesic flow represented by the one-parameter group g_t on $\Gamma\backslash G/B$ [using that the (right) action of B on $\Gamma\backslash G$ commutes with the action of the one-parameter group g_t].

The Liouville measure m on the unit tangent bundle $SV = \Gamma\backslash G/B$ is the measure induced by the Haar measure on G. Let $F\colon \Gamma\backslash G/B \to \mathbb{C}$ be a C^∞ function; then we denote the *correlation function* for F by

$$\rho(t) = \int F\phi_t \cdot F\, dm - \left(\int F\, dm\right)^2 \tag{1.1}$$

2. UNITARY REPRESENTATION THEORY

We shall want to make use of the very explicit knowledge of the irreducible unitary representations of $SL(2, \mathbb{C})$ (by work of Gelfand and Naimark; see ref. 4 for details) to study the behavior of the expression (1.1).

To proceed we need to recall a few basic facts from the theory of unitary representations. To each $g \in G$ we can associate a unitary operator $U_g\colon L^2(\Gamma\backslash G) \to L^2(\Gamma\backslash G)$ on the Hilbert space of square-integrable functions by $(U_g f)(x) = f(gx)$, where $f \in L^2(\Gamma\backslash G)$, $x \in \Gamma\backslash G$.

If $\mathscr{U} = \mathscr{U}(L^2(\Gamma\backslash G))$ denotes the group of unitary operators from $L^2(\Gamma\backslash G)$ to itself, then the map $g \mapsto U_g \in \mathscr{U}$, $g \in G$, is called the *canonical representation* of G.

The canonical representation has a decomposition into a denumerable family of (irreducible unitary) representations of G. More precisely, there exists an orthogonal splitting $L^2(\Gamma\backslash G) = \bigoplus_{i=0}^{+\infty} H_i$ into Hilbert spaces H_i invariant under U_g, for each $g \in G$ (ref. 2, p. 18).

Any function $F \in C^{\infty}(\Gamma\backslash G/B)$ lifts to a function in $C^{\infty}(\Gamma\backslash G)$. If we consider the corresponding decomposition $F(v) = \sum_{i=0}^{+\infty} c_i \cdot F_i(v)$, with $\|F\|_2 = 1$, then we can rewrite (1.1) as

$$\rho(t) = \sum_{i=0}^{+\infty} |c_i|^2 \int F_i(\phi_t v) \cdot F_i(v)\, d(\mathrm{Vol})(v) \tag{2.1}$$

The irreducible unitary representations of $SL(2, \mathbb{C})$ are of the following two types:

2.1. Principal Series for $\rho \in \mathbb{R}$, $n \in \mathbb{Z}$

Let $H = L^2(\mathbb{C}, dx\, dy)$ and the action of G be defined by

$$U_g^{(\rho)}(x) = f\left(\frac{ax+b}{cz+d}\right) \cdot |cz+d|^{n+i\rho-2} \cdot (cz+d)^{-n}, \quad \text{where } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{C})$$

In particular,

$$U_{g_t}^{(\rho)}(x) = f(e^{2t}x)\, e^{(-2+i\rho n)t}$$

2.2. Complementary Series for $0 < \rho < 2$

Let $H \subseteq L^2(\mathbb{C}, dx\, dy)$ be the completion of continuous functions of compact support relative to the inner product $\langle f, g \rangle = \int \hat{f}(z) \cdot \hat{g}(z) \cdot |z|^{2\rho}\, dx\, dy$ and define the action of G by

$$U_{g_t}^{(\rho)} \hat{f}(z) = \hat{f}\left(\frac{ax+b}{cz+d}\right) \cdot |cz+d|^{-2-\rho}, \qquad \text{where } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{C})$$

and we denote $\hat{f}(z) = \int_{-\infty}^{+\infty} f(w) \cdot e^{izw}\, dw$ as the usual Fourier transform on $\mathbb{C}$.[2,4] In particular, $U_{g_t}^{(\rho)} \hat{f}\, (e^{2t}z) \cdot z^{(2-\rho)t}$.

If we are only interested in functions on $V = \Gamma\backslash G/K$ (corresponding to functions on $\Gamma\backslash G$ invariant under the action of K), then: The principal series for $n = 0$ and complementary series are the only irreducible representations containing vectors invariant under the action of the maximal compact subgroup $K = SU(2)$, which when normalized take the following form:

(a) For the principal series ($n = 0$) we can take

$$f(z) = \frac{1}{\sqrt{\pi}} (|z|^2 + 1)^{-(2-i\rho)/2}$$

(b) For the complementary series we take

$$f(z)=\frac{1}{\sqrt{\pi}}(|z|^2+1)^{-(2+\rho)/2}$$

This means that for any function in $L^2(\Gamma\backslash G/K)\subset L^2(\Gamma\backslash G)$ the coefficients corresponding to the principal series for $n\neq 0$ are identically zero.

For each H_i there corresponds an irreducible representation of the above form and an isometry $V\colon H_i\to H$ satisfying $VU_g^{(\rho)}=UV$. In particular, we have that

$$\rho(t)=\sum_{i=0}^{+\infty}|c_i|^2\langle F_i\circ\phi_t, F_i\rangle=\sum_{i=0}^{+\infty}|c_i|^2\langle U_{g_t}^{(\rho)}(VF_i), (VF_i)\rangle \tag{2.2}$$

Remark. The function (VF_i) can readily be identified with eigenfunctions for the Laplace–Beltrami operator on $\Gamma\backslash G/B$, and the decomposition corresponds to an eigenfunction expansion.

We can consider each term separately.

(i) *Principal series with* $n=0$: If (VF_i) is equal to the (normalized) function $f(z)=(1/\sqrt{\pi})(|z|^2+1)^{-(2+i\rho)/2}\in H$, then we could explicitly compute

$$\begin{aligned}\rho(t)&=\langle U_{g_t}^{(\rho)}f, f\rangle\\&=\frac{1}{\pi}\int_{-\infty}^{+\infty}[(|z|^2+1)^{-(2+i\rho)/2}][e^{(-2+i\rho)t}(e^{4t}|z|^2+1)^{-(2-i\rho)/2}]\,dz\\&=\frac{e^{(-2+i\rho)t}}{\pi}\int_0^{2\pi}\int_0^{+\infty}(r^2+1)^{-(2+i\rho)/2}(e^{4t}r^2+1)^{-(2-i\rho)/2}r\,dr\,d\theta\end{aligned}$$

For $\rho\neq 0$ this integral is evaluated with the solution

$$\rho(t)=\frac{2}{\rho}\frac{\sin(\rho t)}{\sinh(2t)}$$

(ii) *Complementary series*: If we assume that (VF_i) is equal to the normalized function $f(z)=(1/\sqrt{\pi})(|z|^2+1)^{-(2+\rho)/2}$, then we can similarly calculate

$$\rho(t)=\frac{2}{\rho}\frac{\sinh(\rho t)}{\sinh(2t)}$$

Remark. For the reader's convenience, we shall briefly comment on the related analysis for $SL(2, \mathbb{R})$ in ref. 1, where the irreducible unitary representations can be divided into the principal, complementary, and discrete series. The authors derive an estimate for the contribution to $\rho(t)$ from the complementary series. However, they omit details for the other two cases and, in point of fact, their claimed estimate on the contribution from the principal series proves to be too optimistic.

3. RATES OF MIXING

We want to add up the contributions to $\rho(t)$ in (2.2) from each of the terms using the estimates in the previous section. We shall assume that the function $F: V \to \mathbb{C}$ is *any square-integrable function constant on fibers of the unit tangent bundle.* We can then identify the images (VF_i) in each of the irreducible representations with the canonical K-fixed functions $f(z)$ and directly apply the estimates from the end of the last section. In particular, we get the following:

$$\rho(t) = \sum_{\substack{\text{principal}\\ \text{series}}} |c_i|^2 \left(\frac{2}{\rho}\frac{\sin(\rho t)}{\sinh(2t)}\right) + \sum_{\substack{\text{comp.}\\ \text{series}}} |c_i|^2 \left(\frac{2}{\rho}\frac{\sinh(\rho t)}{\sinh(2t)}\right) \tag{3.1}$$

where $\sum_{i=0}^{+\infty} |c_i|^2 \times \|F\|_2 < +\infty$ (and, of course, the value ρ depends on the index of summation). By definition of $\rho(t)$, the coefficients corresponding to $\rho = 0$ in the principal series is positive.

To finish we need the following lemma:

Lemma. There are only finitely many terms in the complementary series (corresponding to "small" eigenvalues of the Laplacian).

This easily follows, for example, from the discussion on p. 178 of ref. 3, where Moore observes that the complementary series contains K-fixed vectors corresponding to small eigenvectors of the Laplacian, and the fact that eigenvalues of the Laplacian tend to infinity.

These considerations allow us to get asymptotic estimates on $\rho(t)$ in the case where we restrict to functions F on the unit tangent bundle SV which are constant on fibers (these being functions for which the *only* nontrivial coefficients occurring in the decomposition are for the spherical series—and thus the estimates in Section 2 apply). In particular, we have the following result.

Theorem. There exist a finite set of constants $0 < \alpha_i < 2$ $i = 1,..., n,$ say) such that whenever $F: V \to \mathbb{C}$ (considered as a function $F: SV \to \mathbb{R}$

depending only on the base point on V), then there exist coefficients $C_i > 0$ such that

$$\rho(t) = \sum_{i=1}^{n} C_i e^{-\alpha_i t} + C_0(1+t)e^{-2t} + O(e^{-2t})$$

By the preceding lemma we have the following result.

Corollary. For all but a finite-dimensional space in $L^2(V)$ we have $\rho(t) = O(e^{-2t})$.

Remarks. (i) According to Sinai,[6] if we want to allow the function F to vary on the fibers of the unit tangent bundle $SV = \Gamma\backslash G/B$, then we can use that the fibers are diffeomorphic to the standard two-sphere S^2 [and we can identify $C^\infty(SV) = \Gamma^\infty(V, C^\infty(S^\infty))$]; then on each fiber $S_x V$ over a point $x \in V$ the restriction $F: S_x V \to \mathbb{C}$ can be decomposed into spherical harmonics $\{\Theta_\alpha(\theta): \theta \in S^2, \alpha \in \mathbb{Z}\}$, i.e., $F(x, \theta) = \Sigma_\alpha c_\alpha(x) \cdot \Theta_\alpha(\theta)$. Unfortunately, there seem to be some complications with Sinai's argument (on the last few lines of p. 985).

(ii) I am grateful to a referee for the following observation: In the case of a more general Lie group G of split rank one [e.g., $SO(n, 1)$, $SU(n, 1)$, $Sp(n, 1)$] the contribution of the complimentary series would be $C_\rho e^{(\rho - h)t} + O(e^{(-1+\rho-h)t})$ and the contribution of the principal series would be $\leqslant (1+t)e^{-ht}$.

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